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Approximation Algorithms for Distance-2 Edge Coloring

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1 Introduction

A *radio network* consists of a group of *transceivers* communicating over a common broadcast radio channel. Examples include packet radio networks, cellular phone networks, and satellite networks. Each transceiver has a range (a geographic region) within which it can communicate with other transceivers.

In [3, 5], the authors show that the *link scheduling* problem, which is to assign schedules to the links between transceivers that avoid *primary* and *secondary* interferences, corresponds to the *distance-2 edge coloring* problem in the graph that models the radio network. This undirected graph has one node for each transceiver; for each pair of transceivers such that one transceiver is in the range of the other, there is an edge between the corresponding nodes. In this graph, a valid distance-2 edge coloring must assign distinct colors to any pair of edges between which there is a path of length at most 2. Since each color corresponds to a channel, it is important to produce a distance-2 edge coloring that uses a minimum number of colors.

In a recent result, Erickson et al. [1] showed that, unlike proper edge coloring, the distance-2 edge coloring problem is NP-complete even for bipartite graphs. This raises the need for approximation algorithms.

Preliminaries   We assume $G = (V, E)$ is a connected undirected graph with n vertices and m edges. Let $d_G(u)$ denote the degree of a vertex u in G; let δ and Δ denote the minimum and maximum vertex degree of G respectively. Any two edges $e = (u, v)$ and $e' = (u', v')$ are within distance 2 of each other if and only if either they are adjacent or they are both adjacent to some other edge e''.

A *coloring* of the edges of G with k colors is a mapping $E \to \{1, 2, \ldots, k\}$. A *distance-2 edge coloring* is a coloring of the edges of G such that any two edges within distance 2 are assigned distinct colors. The *distance-2 edge coloring problem* (D2EC) is to find a distance-2 edge coloring with the fewest colors. The minimum number of colors in a distance-2 edge coloring of G is denoted by $\chi'_2(G)$.

2 Approximation algorithms

Since a distance-2 edge coloring is also a proper edge coloring, $\chi'_2(G)$ is at least the chromatic index of G. In addition, if $U \subseteq V$ induces a clique, then $\chi'_2(G) \geq |\{e = (u, v) : u \in U \text{ or } v \in V\}|$. In particular, $\chi'_2(G) \geq \max_{(u,v) \in E} d_G(u) + d_G(v) - 1 \geq \Delta + \delta - 1$.

A greedy algorithm for distance-2 edge coloring of G is to color the edges of G in some order e_1, e_2, ..., e_m using for the edge e_i the color of smallest index not used on any earlier edges within distance 2 of e_i. The key is finding a good ordering of the edges. If the given graph G is a tree, we can color it optimally in linear time by considering the edges in a BFS order.

2.1 c-inductive graphs

We say G is c-inductive (or c-degenerate) if G has a vertex u of degree at most c and $G - u$ is c-inductive. Note that Erickson et al. [1] show that D2EC is NP-hard even for 2-inductive graphs.

Theorem 2.1. *There exists an efficient algorithm to compute an $O(c)$-approximation for c-inductive graphs.*

Proof Sketch: Call a vertex u *light* in G if $|\{v : v \in N_G(u) \text{ and } d_G(v) > c\}| \leq c$. Call an edge (u, v) *light* in G if both u and v are light in G. We can show by an inductive argument that there exists a light edge $e = (u, v)$ in every c-inductive graph. Sort the edges of G in order e_1, e_2, ..., e_m such that e_i is light in $G_{i-1} = G - \bigcup_{1 \leq j \leq i-1} e_j$. Color the edges greedily in the reverse order.

We can prove that if e_i is light in G_{i-1}, then there are at most $4c\Delta - 2c^2$ edges within distance 2 of e_i in G that are colored before e_i. The greedy algorithm uses at most one color more so that the total number of colors used is $O(c)$ times the lower bound of Δ. □

Since every planar graph is 5-inductive, the algorithm yields a $O(1)$-approximation for planar graphs.

2.2 Unit disk graphs

Given a set V of points in the plane, the unit disk graph $G = (V, E)$ is formed by putting an edge between u and v if and only if $\text{dist}(u, v) \leq 1$. In other words, if D_u and D_v are disks of radius 1 centered at u and v respectively, then $(u, v) \in E$ if and only if $v \in D_u$ (equivalently, $u \in D_v$).

Theorem 2.2. *There exists a greedy algorithm for 5-approximate distance-2 edge coloring of the edges in a unit disk graph.*

Proof Sketch: Consider the points in V in lexicographic order $\prec$ of their y-coordinates. The lexicographic order $\prec$ on the edges of G is such that for any two edges $e_i = (u_i, v_i)$ and $e_j = (u_j, v_j)$ with $u_i \prec v_i$ and $u_j \prec v_j$, we have $e_i \prec e_j$ whenever either $u_i \prec u_j$, or $u_i = u_j$ and $v_i \prec v_j$.

Any edge $e' = (u', v')$ that is within distance 2 of the edge $e = (u, v)$ must have at least one endpoint in $D_u \cup D_v$. Furthermore, if the edge e' is already colored, then at least one of u' and v' must precede u in the lexicographic ordering. Therefore, at least one endpoint of e' must lie in the region L that is the portion of $D_u \cup D_v$ below the line uv. It is easy to see that L can be covered by at most 5 unit-diameter disks. All the vertices in any one of these disks form a clique and any two edges with at least one endpoint incident on one of these cliques must get distinct colors. Therefore, the number of edges within distance 2 of any given edge that have already been colored is at most $5\chi'_2(G)$. The greedy algorithm uses at most one more color. $\quad\square$

2.3 (r, s)-civilized graphs

For each fixed pair of real values $r, s > 0$, an (r, s)-*civilized graph* [6] is a graph that can be embedded in the plane such that the length of each edge is at most r and the distance between any two adjacent vertices is at least s. Such graphs model radio networks with occlusions. We will assume that the planar embedding of the graph is given as input. Note that the maximum degree of any vertex in G is $O(r^2/s^2)$.

Theorem 2.3. *There exists an efficient algorithm for distance-2 edge coloring of (r, s)-civilized graphs using at most twice the minimum number of colors.*

Proof Sketch: Using ideas in [2], we can partition the given graph G into levels such that the subgraphs induced by each level have constant treewidth.

We can prove that if H is a graph of constant treewidth and bounded degree, then a distance-2 edge coloring of H can be obtained in polynomial time. In fact even if some of the edges of H are colored *a priori* then the coloring can be extended to the remaining edges optimally (only with respect to the remaining edges) in polynomial time.

We can thus color each subgraph optimally. Subgraphs induced by successive levels share vertices but the colors used on the subgraph at level i can be reused on the subgraph at level $i+2$. Therefore, we obtain a 2-approximation by this level-wise coloring algorithm. $\quad\square$

3 Experimental results

We implemented a simple randomized distributed algorithm where each transceiver u independently picks a random subset of $d(u)$ channels out of a fixed global palette and uses this subset of channels to communicate to its neighbors. We tested this algorithm on random graphs in the $G(n, p)$ model and random congruent disk graphs. If two edges within distance 2 were assigned the same color by the random choice, then one of the edges was uncolored. We measured the fraction of the edges that were colored without distance-2 conflicts. This represents one round of the algorithm. In subsequent rounds, only edges that have not been colored in previous rounds were assigned a random color chosen as before from the same palette. We observed that if the global palette contains enough colors (close to the upper bound of Δ^2), then a large fraction (75% and above) of the edges are successfully colored in one round and almost all edges are colored after 10 rounds. If the palette size is decreased to $\Delta \log \Delta$, then only a small fraction of edges is successfully colored in the first round. As expected, the performance of the algorithm degrades with increase in edge probability p for random graphs and with increase in the radius for disk graphs: both imply denser graphs. One reason is that the probability of assigning a valid color to an edge decreases as its neighborhood gets larger. Thus, if there exists an *a priori* bound of Δ on the maximum number of outgoing transmissions per node in the network, then the total number of channels can be fixed at Δ^2 and a large fraction of the total pairwise communications can be successfully completed via the randomized distributed algorithm. The running time of the algorithm in such cases is quite small.

4 Concluding remarks

We also have algorithms that achieve a 2-approximation for outerplanar graphs and a $O(\min\{n/D, n/g\})$-approximation for general graphs where D is the diameter and g is the girth. We are investigating several open problems including the hardness of approximating D2EC and finding efficient distributed approximation algorithms.

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